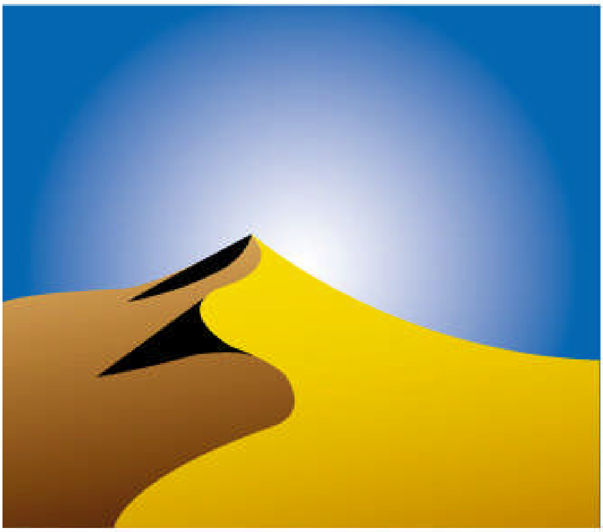
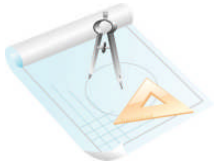
積分の計算（数Ⅲ）



重要例題集　積分の計算



第１問【不定積分１】

次の不定積分を求めよ．

(1) $\displaystyle\int \frac{(x+1)^2}{x-1}dx$　　(2) $\displaystyle\int \frac{x^2+3x+4}{x+2}dx$　　(3) $\displaystyle\int \frac{x^3}{x-1}dx$

＜解＞

(1) $\displaystyle\int \frac{(x+1)^2}{x-1}dx=\int\left(x+3+\frac{4}{x-1}\right)dx=\frac{1}{2}x^2+3x+4\log|x-1|+C$

(2) $\displaystyle\int \frac{x^2+3x+4}{x+2}dx=\int\left(x+1+\frac{2}{x+2}\right)dx=\frac{1}{2}x^2+x+2\log|x+2|+C$

(3) $\displaystyle\int \frac{x^3}{x-1}dx=\int\left(x^2+x+1+\frac{1}{x-1}\right)dx=\frac{1}{3}x^3+\frac{1}{2}x^2+x+\log|x-1|+C$

（Cは積分定数）

第２問【不定積分２】

次の不定積分を求めよ．

(1) $\displaystyle\int (e^x+1)^2dx$　　(2) $\displaystyle\int (e^x-e^{-x})^3dx$

＜解＞

(1) $\displaystyle\int (e^x+1)^2dx=\int(e^{2x}+2e^x+1)dx=\frac{1}{2}e^{2x}+2e^x+x+C$

(2) $\displaystyle\int (e^x-e^{-x})^3dx=\int(e^{3x}-3e^x+3e^{-x}-e^{-3x})dx=\frac{1}{3}e^{3x}-3e^x-3e^{-x}+\frac{1}{3}e^{-3x}+C$

第 3 問【部分分数分解と不定積分 1】

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{dx}{x(x+2)}$ (2) $\displaystyle\int \frac{dx}{x^2-4}$ (3) $\displaystyle\int \frac{x}{(x-1)(2x-1)}dx$

＜解＞

(1) $\displaystyle\int \frac{dx}{x(x+2)} = \int \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \frac{1}{2}(\log|x| - \log|x+2|) + C = \frac{1}{2}\log\left|\frac{x}{x+2}\right| + C$

(2) $\displaystyle\int \frac{dx}{x^2-4} = \int \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx = \frac{1}{4}(\log|x-2| - \log|x+2|) + C$

$\displaystyle = \frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + C$

(3) $\displaystyle\int \frac{x}{(x-1)(2x-1)}dx = \int \left(\frac{1}{x-1} - \frac{1}{2x-1}\right)dx = \log|x-1| - \frac{1}{2}\log|2x-1| + C$

$\displaystyle = \frac{1}{2}\log\frac{(x-1)^2}{|2x-1|} + C$

第 4 問【部分分数分解と不定積分 2】

(1) 次の等式が成り立つように, 定数 a, b, c の値を定めよ.

$$\frac{3x+2}{x(x+1)^2} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2}$$

(2) $\displaystyle\int \frac{3x+2}{x(x+1)^2}dx$ を計算せよ.

＜解＞

(1) 等式の両辺に $x(x+1)^2$ を掛けて

$$3x+2 = a(x+1)^2 + bx(x+1) + cx$$

右辺を整理すると

$$3x+2 = (a+b)x^2 + (2a+b+c)x + a$$

これが x についての恒等式であるから

$$a+b=0,\ 2a+b+c=3,\ a=2$$

これを解いて $a=2$, $b=-2$, $c=1$

(2) $\displaystyle\int \frac{3x+2}{x(x+1)^2}dx = \int \left\{\frac{2}{x} - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right\}dx = 2\log|x| - 2\log|x+1| - \frac{1}{x+1} + C$

$\displaystyle = 2\log\left|\frac{x}{x+1}\right| - \frac{1}{x+1} + C$

第 5 問【三角関数と不定積分】

次の不定積分を求めよ.

(1) $\displaystyle\int \sin^2 2x\,dx$　　(2) $\displaystyle\int \sin 2x \sin 4x\,dx$　　(3) $\displaystyle\int \cos 4x \sin x\,dx$

(4) $\displaystyle\int \sin^4 x\,dx$　　(5) $\displaystyle\int (\sin x+\cos x)^4 dx$

＜解＞

(1) $\displaystyle\int \sin^2 2x\,dx=\int \frac{1-\cos 4x}{2}dx=\frac{1}{2}x-\frac{1}{8}\sin 4x+C$

(2) $\sin 2x \sin 4x=-\dfrac{1}{2}(\cos 6x-\cos 2x)$ であるから

$$\int \sin 2x \sin 4x\,dx=-\frac{1}{2}\int(\cos 6x-\cos 2x)dx=-\frac{1}{12}\sin 6x+\frac{1}{4}\sin 2x+C$$

(3) $\cos 4x \sin x=\dfrac{1}{2}(\sin 5x-\sin 3x)$ であるから

$$\int \cos 4x \sin x\,dx=\frac{1}{2}\int(\sin 5x-\sin 3x)dx=-\frac{1}{10}\cos 5x+\frac{1}{6}\cos 3x+C$$

(4) $\sin^4 x=(\sin^2 x)^2=\left(\dfrac{1-\cos 2x}{2}\right)^2=\dfrac{1}{4}(1-2\cos 2x+\cos^2 2x)$

$$=\frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)=\frac{1}{8}(3-4\cos 2x+\cos 4x)$$

よって　$\displaystyle\int \sin^4 x\,dx=\frac{1}{8}\int(3-4\cos 2x+\cos 4x)dx=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$

(5) $(\sin x+\cos x)^4=(1+2\sin x\cos x)^2=(1+\sin 2x)^2=1+2\sin 2x+\sin^2 2x$

$$=1+2\sin 2x+\frac{1-\cos 4x}{2}=\frac{3}{2}+2\sin 2x-\frac{1}{2}\cos 4x$$

よって　$\displaystyle\int(\sin x+\cos x)^4 dx=\int\left(\frac{3}{2}+2\sin 2x-\frac{1}{2}\cos 4x\right)dx$

$$=\frac{3}{2}x-\cos 2x-\frac{1}{8}\sin 4x+C$$

第 6 問【置換積分 1】

次の定積分を求めよ．

(1) $\displaystyle\int_0^2 2x(x^2+1)^3dx$　(2) $\displaystyle\int_1^2 \frac{x^2-2x}{x^3-3x^2+1}dx$　(3) $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+4}}dx$

(4) $\displaystyle\int_1^e \frac{\log x}{x}dx$　(5) $\displaystyle\int_0^1 x^2e^{x^3}dx$　(6) $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 x\cos^3 x\,dx$

＜解＞

(1) $x^2+1=t$ とおくと　$2xdx=dt$

x	$0 \longrightarrow 2$
t	$1 \longrightarrow 5$

$$\int_0^2 2x(x^2+1)^3dx=\int_1^5 t^3dt=\left[\frac{t^4}{4}\right]_1^5=156$$

別解 $$\int_0^2 2x(x^2+1)^3dx=\int_0^2 (x^2+1)^3(x^2+1)'dx=\left[\frac{1}{4}(x^2+1)^4\right]_0^2=156$$

(2) $x^3-3x^2+1=t$ とおくと　$3(x^2-2x)dx=dt$

x	$1 \longrightarrow 2$
t	$-1 \longrightarrow -3$

$$\int_1^2 \frac{x^2-2x}{x^3-3x^2+1}dx=\int_{-1}^{-3}\frac{1}{3}\cdot\frac{1}{t}dt=\frac{1}{3}\Big[\log|t|\Big]_{-1}^{-3}=\frac{1}{3}\log 3$$

別解 $$\int_1^2 \frac{x^2-2x}{x^3-3x^2+1}dx=\int_1^2 \frac{(x^3-3x^2+1)'}{x^3-3x^2+1}\cdot\frac{1}{3}dx=\frac{1}{3}\Big[\log|x^3-3x^2+1|\Big]_1^2=\frac{1}{3}\log 3$$

(3) $\sqrt{x^2+4}=t$ とおくと　$x^2+4=t^2$ から　$2xdx=2tdt$

x	$0 \longrightarrow 1$
t	$2 \longrightarrow \sqrt{5}$

$$\int_0^1 \frac{x}{\sqrt{x^2+4}}dx=\int_2^{\sqrt{5}}\frac{1}{t}\cdot tdt=\Big[t\Big]_2^{\sqrt{5}}=\sqrt{5}-2$$

別解 $$\int_0^1 \frac{x}{\sqrt{x^2+4}}dx=\int_0^1 \frac{(x^2+4)'}{\sqrt{x^2+4}}\cdot\frac{1}{2}dx=\Big[(x^2+4)^{\frac{1}{2}}\Big]_0^1=\sqrt{5}-2$$

(4) $\log x = t$ とおくと　$\dfrac{dx}{x} = dt$

x	$1 \longrightarrow e$
t	$0 \longrightarrow 1$

$$\int_1^e \frac{\log x}{x}dx = \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}$$

別解 $$\int_1^e \frac{\log x}{x}dx = \int_1^e (\log x)(\log x)'dx = \left[\frac{1}{2}(\log x)^2\right]_1^e = \frac{1}{2}$$

(5) $x^3 = t$ とおくと　$3x^2dx = dt$

x	$0 \longrightarrow 1$
t	$0 \longrightarrow 1$

$$\int_0^1 x^2e^{x^3}dx = \int_0^1 \frac{1}{3}e^t dt = \left[\frac{1}{3}e^t\right]_0^1 = \frac{1}{3}(e-1)$$

別解 $$\int_0^1 x^2e^{x^3}dx = \int_0^1 \left\{e^{x^3}(x^3)'\cdot\frac{1}{3}\right\}dx = \left[\frac{1}{3}e^{x^3}\right]_0^1 = \frac{1}{3}(e-1)$$

(6) $\sin x = t$ とおくと　$\cos x\,dx = dt$

x	$0 \longrightarrow \dfrac{\pi}{6}$
t	$0 \longrightarrow \dfrac{1}{2}$

$$\int_0^{\frac{\pi}{6}} \sin^2 x\cos^3 x\,dx = \int_0^{\frac{1}{2}} t^2(1-t^2)dt$$

$$= \int_0^{\frac{1}{2}} (t^2 - t^4)dt = \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^{\frac{1}{2}} = \frac{17}{480}$$

別解 $$\int_0^{\frac{\pi}{6}} \sin^2 x\cos^3 x\,dx = \int_0^{\frac{\pi}{6}} \sin^2 x(1-\sin^2 x)(\sin x)'dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{6}} = \frac{17}{480}$$

第 7 問【置換積分２】

次の定積分を求めよ.

(1) $\displaystyle\int_0^1 \frac{dx}{(1+2x)^3}$　(2) $\displaystyle\int_0^1 \frac{2e^x}{e^x+1}dx$　(3) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}dx$

(1) $\underline{1+2x=t}$ とおくと　$2dx=dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 3$

$$\int_0^1 \frac{dx}{(1+2x)^3} = \int_1^3 \frac{dt}{2t^3} = \left[-\frac{1}{4t^2}\right]_1^3 = \frac{2}{9}$$

別解 $$\int_0^1 \frac{dx}{(1+2x)^3} = \left[-\frac{1}{2}\cdot\frac{1}{(1+2x)^2}\cdot\frac{1}{2}\right]_0^1 = \frac{2}{9}$$

(2) $\underline{e^x+1=t}$ とおくと　$e^x dx=dt$

x	$0 \longrightarrow 1$
t	$2 \longrightarrow e+1$

$$\int_0^1 \frac{2e^x}{e^x+1}dx = \int_2^{e+1} \frac{2}{t}dt = 2\Big[\log t\Big]_2^{e+1} = 2\log\frac{e+1}{2}$$

別解 $$\int_0^1 \frac{2e^x}{e^x+1}dx = \int_0^1 \frac{2(e^x+1)'}{e^x+1}dx = 2\Big[\log(e^x+1)\Big]_0^1 = 2\log\frac{e+1}{2}$$

(3) $\underline{2+\cos x=t}$ とおくと　$-\sin x\,dx=dt$

x	$0 \longrightarrow \frac{\pi}{2}$
t	$3 \longrightarrow 2$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}dx = \int_3^2 \left(\frac{-1}{t}\right)dt = -\Big[\log|t|\Big]_3^2 = \log\frac{3}{2}$$

別解 $$\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}dx = \int_0^{\frac{\pi}{2}} \left\{\frac{-(2+\cos x)'}{2+\cos x}\right\}dx = -\Big[\log(2+\cos x)\Big]_0^{\frac{\pi}{2}} = \log\frac{3}{2}$$

第 8 問【特別な定積分】

次の定積分を求めよ.

(1) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$　(2) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}$　(3) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{8-2x^2}}$

(4) $\displaystyle\int_0^2 \frac{dx}{x^2+4}$　(5) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{x^2+3}$　(6) $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{3x^2+12}$

＜解＞

(1) $x=3\sin\theta$ とおくと　$dx=3\cos\theta\, d\theta$

x	$0 \longrightarrow 3$
θ	$0 \longrightarrow \frac{\pi}{2}$

$0 \leqq \theta \leqq \frac{\pi}{2}$ では　$\cos\theta \geqq 0$ であるから

$$\sqrt{9-x^2}=\sqrt{9(1-\sin^2\theta)}=3\sqrt{\cos^2\theta}=3\cos\theta$$

よって　$\displaystyle\int_0^3 \sqrt{9-x^2}\,dx=\int_0^{\frac{\pi}{2}} 3\cos\theta\cdot 3\cos\theta\, d\theta=9\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta=9\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta$

$$=\frac{9}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{9}{4}\pi$$

(2) $x=4\sin\theta$ とおくと　$dx=4\cos\theta\, d\theta$

x	$0 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{6}$

$0 \leqq \theta \leqq \frac{\pi}{6}$ では　$\cos\theta \geqq 0$ であるから

$$\sqrt{16-x^2}=4\sqrt{\cos^2\theta}=4\cos\theta$$

よって　$\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}=\int_0^{\frac{\pi}{6}}\frac{4\cos\theta}{4\cos\theta}d\theta=\int_0^{\frac{\pi}{6}}d\theta=\Big[\theta\Big]_0^{\frac{\pi}{6}}=\frac{\pi}{6}$

(3) $x=2\sin\theta$ とおくと　$dx=2\cos\theta\, d\theta$

x	$0 \longrightarrow \sqrt{3}$
θ	$0 \longrightarrow \frac{\pi}{3}$

$0 \leqq \theta \leqq \frac{\pi}{3}$ では　$\cos\theta \geqq 0$ であるから

$$\sqrt{8-2x^2}=2\sqrt{2}\cdot\sqrt{\cos^2\theta}=2\sqrt{2}\cos\theta$$

よって　$\displaystyle\int_0^{\sqrt{3}}\frac{dx}{\sqrt{8-2x^2}}=\int_0^{\frac{\pi}{3}}\frac{2\cos\theta}{2\sqrt{2}\cos\theta}d\theta=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}d\theta=\frac{1}{\sqrt{2}}\Big[\theta\Big]_0^{\frac{\pi}{3}}=\frac{\sqrt{2}}{6}\pi$

(4) $x=2\tan\theta$ とおくと　$dx=\dfrac{2}{\cos^2\theta}d\theta$

x	$0 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{4}$

$$\int_0^2\frac{dx}{x^2+4}=\int_0^{\frac{\pi}{4}}\frac{1}{4(\tan^2\theta+1)}\cdot\frac{2}{\cos^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}d\theta=\frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{4}}=\frac{\pi}{8}$$

(5) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{x^2+3}=2\int_0^{\sqrt{3}}\frac{dx}{x^2+3}$

$x=\sqrt{3}\tan\theta$ とおくと　$dx=\dfrac{\sqrt{3}}{\cos^2\theta}d\theta$

x	$0 \longrightarrow \sqrt{3}$
θ	$0 \longrightarrow \frac{\pi}{3}$

$$\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{x^2+3}=2\int_0^{\frac{\pi}{4}}\frac{1}{3(\tan^2\theta+1)}\cdot\frac{\sqrt{3}}{\cos^2\theta}d\theta$$

$$=\frac{2\sqrt{3}}{3}\int_0^{\frac{\pi}{4}}d\theta=\frac{2\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{4}}=\frac{\sqrt{3}}{6}\pi$$

(6) $x=2\tan\theta$ とおくと　$dx=\dfrac{2}{\cos^2\theta}d\theta$

x	$0 \longrightarrow 2\sqrt{3}$
θ	$0 \longrightarrow \frac{\pi}{3}$

$$\int_0^{2\sqrt{3}}\frac{dx}{3x^2+12}=\int_0^{\frac{\pi}{3}}\frac{1}{12(\tan^2\theta+1)}\cdot\frac{2}{\cos^2\theta}d\theta$$

$$=\frac{1}{6}\int_0^{\frac{\pi}{3}}d\theta=\frac{1}{6}\Big[\theta\Big]_0^{\frac{\pi}{3}}=\frac{\pi}{18}$$

第 9 問【部分積分 1】

次の定積分を求めよ.

(1) $\displaystyle\int_0^1 x(x-1)^4dx$ (2) $\displaystyle\int_0^{\frac{\pi}{2}}(x+2)\cos x\,dx$ (3) $\displaystyle\int_1^2 x^4\log x\,dx$

＜解＞

(1) $$\int_0^1 x(x-1)^4dx=\left[x\cdot\frac{(x-1)^5}{5}\right]_0^1-\int_0^1\frac{(x-1)^5}{5}dx$$

$$=0-\frac{1}{5\cdot6}\left[(x-1)^6\right]_0^1=\frac{1}{30}$$

(2) $$\int_0^{\frac{\pi}{2}}(x+2)\cos x\,dx=\left[(x+2)\sin x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x\,dx$$

$$=\left(\frac{\pi}{2}+2\right)+\left[\cos x\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}+1$$

(3) $$\int_1^2 x^4\log x\,dx=\left[\frac{x^5}{5}\log x\right]_1^2-\int_1^2\frac{x^4}{5}dx=\frac{32}{5}\log 2-\left[\frac{x^5}{25}\right]_1^2=\frac{32}{5}\log 2-\frac{31}{25}$$

第 10 問【部分積分 2】

次の定積分を求めよ.

(1) $\displaystyle\int_0^{\frac{\pi}{2}}e^x\cos x\,dx$ (2) $\displaystyle\int_0^{\frac{\pi}{2}}e^{-3x}\sin x\,dx$

＜解＞

(1) $$\int_0^{\frac{\pi}{2}}e^x\cos x\,dx=\left[e^x\cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}e^x\sin x\,dx=-1+\left[e^x\sin x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}e^x\cos x\,dx$$

よって $\displaystyle 2\int_0^{\frac{\pi}{2}}e^x\cos x\,dx=e^{\frac{\pi}{2}}-1$ ゆえに $\displaystyle\int_0^{\frac{\pi}{2}}e^x\cos x\,dx=\frac{e^{\frac{\pi}{2}}-1}{2}$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-3x}\sin x\,dx=\left[-\frac{1}{3}e^{-3x}\sin x\right]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}} e^{-3x}\cos x\,dx$

$$=-\frac{1}{3}e^{-\frac{3}{2}\pi}-\left[\frac{1}{9}e^{-3x}\cos x\right]_0^{\frac{\pi}{2}}-\frac{1}{9}\int_0^{\frac{\pi}{2}} e^{-3x}\sin x\,dx$$

よって　$\displaystyle\frac{10}{9}\int_0^{\frac{\pi}{2}} e^{-3x}\sin x\,dx=\frac{1}{9}\left(1-3e^{-\frac{3}{2}\pi}\right)$

ゆえに　$\displaystyle\int_0^{\frac{\pi}{2}} e^{-3x}\sin x\,dx=\frac{1}{10}\left(1-3e^{-\frac{3}{2}\pi}\right)$

第 11 問【部分積分と漸化式】

0 または正の整数 n に対して，$I_n=\displaystyle\int_0^1(1-x^2)^n dx$ とおくとき，$I_n=\dfrac{2n}{2n+1}I_{n-1}$　$(n\geqq 1)$ が成り立つことを示せ.

＜解＞

$x=\sin\theta$ とおくと

$$I_n=\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)^n\cos\theta\,d\theta=\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos^{2n}\theta(\sin\theta)'d\theta$$

$$=\Big[\cos^{2n}\theta\sin\theta\Big]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}2n\cos^{2n-1}\theta(-\sin\theta)\sin\theta\,d\theta$$

$$=2n\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta(1-\cos^2\theta)d\theta$$

$$=2n\left(\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\,d\theta-\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta\,d\theta\right)=2n(I_{n-1}-I_n)$$

ゆえに　$I_n=\dfrac{2n}{2n+1}I_{n-1}$　$(n\geqq 1)$